GAUSSIAN PROCESSES
EXERCISE SHEET 6: ENTROPY AND CLT(II)

Exercise 1 (Smoothed convergence implies convergence). *Let $X_1, X_2, \dots$ be random variables of zero mean and unit variance. Let $Y_1, Y_2, \dots$ be i.i.d. standard Gaussians, independent from $X_1, X_2, \dots$ and define $\hat{X}_n = X_n + Y_n$. Suppose that the sequence of variables $\hat{X}_n$ converges in law to a centered Gaussian of variance 2. Prove that X_n converges in law to a standard Gaussian.*

Exercise 2 (CLT using Stein's lemma). *Use Stein's lemma to prove the usual CLT for sums of i.i.d. random variables of finite variance. You may proceed in two steps:*

(1) By showing tightness, prove that there exist subsequences that converge in law.
(2) Characterise the limit using Stein's lemma.

Assume some extra conditions on X_i if it helps!

Exercise 3 (Equivalence of EPI). *Let X and Y be independent random variables with finite differential entropies, and denote by $h(\cdot)$ the differential entropy. Prove that the following statements are equivalent:*

(1) $\exp\big(2h(X+Y)\big) \geq \exp\big(2h(X)\big) + \exp\big(2h(Y)\big)$.
(2) For each $\lambda \in (0,1)$,

$$h\big(\sqrt{\lambda}\, X + \sqrt{1-\lambda}\, Y\big) \geq \lambda\, h(X) + (1-\lambda)\, h(Y).$$

(3) Let X' and Y' be independent Gaussian random variables with the same entropies as X and Y, respectively. Then

$$h(X+Y) \geq h(X'+Y').$$

Exercise 4 (Discretisation of the Ornstein–Uhlenbeck process). *The (stationary) Ornstein–Uhlenbeck SDE is commonly written*

$$dX_t = -X_t\, dt + \sqrt{2}\, dB_t.$$

To make this rigorous via discretisation, consider the time step $\epsilon > 0$ and define a discrete-time process X^ϵ on the grid $\{0, \epsilon, 2\epsilon, \dots\}$ by

$$X^\epsilon(t+\epsilon) - X^\epsilon(t) = -\epsilon\, X^\epsilon(t) + \sqrt{2\epsilon}\, Y_t,$$

where $(Y_t)_{t \in \{0,\epsilon,2\epsilon,\dots\}}$ are i.i.d. standard Gaussian random variables and $X^\epsilon(0) = X_0$ is given.

(a) Show that for $n \in \mathbb{N}$,

$$X^\epsilon(n\epsilon) \;=\; (1-\epsilon)^n X_0 + \sqrt{2\epsilon} \sum_{k=1}^{n} (1-\epsilon)^{n-k} Y_{k\epsilon}.$$

Use this to show that for fixed $t > 0$ and $\epsilon \downarrow 0$ (with $n = \lfloor t/\epsilon \rfloor$) the law of $X^\epsilon(t)$ converges to

$$X_t \overset{d}{=} e^{-t} X_0 + \sqrt{1 - e^{-2t}}\, Z,$$

where $Z \sim \mathcal{N}(0,1)$ is independent of X_0. Furthermore, try to explain why we can write

$$X_t \;=\; e^{-t} X_0 + \sqrt{2} \int_0^t e^{-(t-s)}\, dB_s,$$

(b) Let $p(t,x)$ denote the density of X_t (assume it exists and is smooth). Conclude that p satisfies the equation

$$\frac{\partial}{\partial t}p(t,x) \;=\; \frac{\partial}{\partial x}\big(x\,p(t,x)\big) \;+\; \frac{\partial^2}{\partial x^2}p(t,x),$$

and that,

$$\frac{d}{dt}\mathbb{E}\big[f(X_t)\big] \;=\; \mathbb{E}\big[f''(X_t)\big] - \mathbb{E}\big[X_t f'(X_t)\big].$$

(You can add some suitable conditions on f.)

Exercise 5 (Cramér's theorem (Not Examinable))**.** *Let X and Y be independent random variables such that $X+Y$ is Gaussian. Prove that both X and Y are Gaussian.*